反演變換

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1 Introduction

反演變換對於許多複雜的幾何問題都能發揮它的強大之處

2 定義與性質

Definition 1.
對於兩個點 P、O以及實數 k，若點 P^* 滿足 $OP \cdot OP^* = k$，則稱 P^*為 P 以反演冪 k、反演中心 O產生的像
其中，若 $k > 0$則稱為**雙曲線型變換**，若$k < 0$則是**橢圓型變換**
Remark.注意，邊相乘在這邊是具有方向性的，同向$k > 0$，反之則小於0

接下來，來關心不同的圖形反演後的像會有什麼樣的變化

Property 2.
$i.$一條直線若通過反演中心，則反演後的像為它自己
$ii.$一條直線若不通過反演中心，則反演後的為通過反演中心的一個圓
$iii.$一個通過反演中心的圓，反演後為不通過反演中心的直線

$iv.$一個不通過反演中心的圓，反演後為不通過反演中心的另一圓
(證明的部分可以自己仔細思考看看，原則上這些性質解題都可以直接使用)

Property 3.
A, B經過同一個反演後的像為 A^*, B^*，則四點共圓
Proof.圓冪性質即可說明

Property 4.
A, B經過同一個反演後的像為 A^*, B^*，則 $\overline{A^*B^*} = \frac{|k|}{OA \cdot OB}\overline{AB}$
本結論可以用前一個性質得到相似形的關係解出

Property 5.
反演後兩平滑曲線相交角度保持不變(平滑曲線包含圓形和直線)

Definition 6.
一個點 P對一圓O的兩切點連線稱為P對於圓O(半徑為 r)的極線，其中OP與極線的交點就是P對於中心為 O，冪為 r^2的反演變換像
其中，若 P在圓上，則極線就是它的切線
P.S. 極線也有許多相關知識，有興趣可以上網查詢相關資料(之後或許也會做講義(?))

Theorem 7.(La Hire)
若A在B的極線上，則B在A的極線上(可以嘗試用共圓的性質證明之)

Theorem 8.(曼海姆定理)

設 ΔABC 之外接圓為 Ω，一圓與 AB, AC, Ω相切，則在 AB, AC 上的切點連線中點即為ΔABC的內心

Theorem 9.(旁賽列閉合)

設 ΔABC之外接圓為 Ω_1， 內切圓為 Ω_2。在Ω_1上取一點D，令它對Ω_2的兩切線與Ω_1交點為E, F，則Ω_2為ΔDEF之內切圓

3 例題和習題

3.1 例題

例題 0.

利用反演證明托勒密定理

例題 1.(USAMO 1993 P2)

設四邊形$ABCD$對角線互相垂直且交於E，證明：E關於四個邊的反射點共圓

Solution 1-1:

設它對於四個邊(AB, BC, CD, AD)的垂足為 W, X, Y, Z，利用垂直的角度們可以得$AXEZ, BWEX, CXEY, DYEZ$點組們四點共圓，則可以計算角度：

$\angle ZWX = 180^\circ - (\angle AWZ + \angle BWX) = 180^\circ - (\angle AEZ + \angle BEX)$

$= 180^\circ - (\angle ADE + \angle BCE) = 180^\circ - (\angle ZYE + \angle XYE) = 180^\circ - \angle XYZ$

於是WXYZ四點共圓，又其反射點形成的四邊形即為$WXYZ$對E的兩倍位似，必共圓，得證。

Solution 1-2:

設它對於四個邊(AB, BC, CD, AD)的反射點為 W, X, Y, Z，則有W, Z, E在以A為圓心的圓型上(同理對於其餘四個點組)。 接著考慮他們對E反演後的像，以圓$\odot(WZE)$為例，以E反演後$\odot(WZE)$變成一個不過E的直線，而且它垂直於AE(E過它的圓心所以反演後他們垂直)

於是，反演後W, X, Y, Z的像W^*, X^*, Y, Z^*為四線交點，是一個矩形，四點共圓，而四點變換前就不過E，所以變換後共圓即代表變換前亦共圓

看出來了嗎?其實反演就是將一個圖形變成另一個之後讓你有更多不同的角度進行切入並解答它(雖然上面的例子看似多此一舉...)

例題 2.(NIMO 2014)

ΔABC的內心是 I，ΔDEF為內切三角形，點Q為滿足 $AB \perp QB$，$AC \perp QC$，P是QI與EF的交點。證明：$DP \perp EF$

hint 1.反演中心是I

hint 2.中點$\rightarrow$九點圓

利用反演解題時，你必須能清楚說出每個點的像是什麼再解題

例題 3.(China Western 2006)

設四邊形 $ADBE$內接於以AB為直徑的圓，對角線相交於C，設Γ為 BOD外

接圓，其中O為AB中點，設F是 Γ的點O對其圓心的對稱點(我們稱之為對徑點)，射線FC交 Γ於另一點G。證明：$AOGE$四點共圓。

hint 1.反演中心是O

hint 2.G除了以線與圓的交點表達有沒有更好的取代方式?

hint 3.定義某個點G'為$\odot DOB$與AOE的交點

3.2 習題

習題 1.(USAMO 2009 P5)

Trapezoid $ABCD$, with $\overline{AB}||\overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersects $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

習題 2.(2015 IMO P3)

Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

習題 3.(2015 Canada MO P4)

Let ABC be an acute-angled triangle with circumcenter O. Let I be a circle with center on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that

$$BP \cdot CQ = AP \cdot AQ.$$

Prove that I is tangent to the circumcircle of triangle BOC.

習題 4.(2013 ELMO shortlist)

Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

習題 5.(2010 Croatia MO P7)

Given a non- isosceles triangle ABC. Let the points B' and C' be symmetric to the points B and C wrt AC and AB respectively. If the circles circumscribed around triangles ABB' and ACC' intersect at point P, prove that the line AP passes through the center of the circumcircle of the triangle ABC.

習題 6.(2015 Taiwan TST 3J)

In a scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of triangle AEF at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter ST is orthogonal to the nine-point circle of triangle BIC.

習題 7.(2008 BAMO G4)

A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with the circumcircles of BDC, CDA, and ADB, respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BA_1} + \frac{CD}{CC_1} = 1.$$

習題 8.(2019 IMO P6)

Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

習題 9.(2020 Baltic Way)

Let ABC be a triangle with $AB > AC$. The internal angle bisector of $\angle BAC$ intersects the side BC at D. The circles with diameters BD and CD intersect the circumcircle of $\triangle ABC$ a second time at $P \neq B$ and $Q \neq C$, respectively. The lines PQ and BC intersect at X. Prove that AX is tangent to the circumcircle of $\triangle ABC$.

4 後記

最近...鯨魚心情有點小複雜(?)或許應該嘗試一些競賽以外的事情(?)
反正就是得失心要輕一點Rah